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New Developments in the Ergodic Theory of Nonlinear Dynamical Systems

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New developments in the ergodic theory of nonlinear dynamical systems

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The purpose of this paper is to give a survey of recent results on non-uniformly hyperbolic dynamical systems. The emphasis is on the existence of strange attractors and Sinai–Ruelle–Bowen measures for Hénon maps, but we also describe results about statistical properties of such dynamical systems and state some of the open questions in this area.

1. Introduction

Our aim is to survey some recent developments in the ergodic theory of dynamical systems in dimension one or two, more specifically the theory of chaotic behaviour and strange attractors. There already exist excellent overviews of this subject by Carleson (1991) and Young (1993), and there is a considerable overlap with those articles and the present. We have, however, been able to include some more recent material. For a somewhat older but more complete survey with a lot of background material see Eckmann & Ruelle (1985).

We basically have two model problems in mind.

1. *The Lorenz equations*. This is a flow in $\mathbb{R}^3$ generated by the following system of nonlinear differential equations:

$$\begin{aligned} \dot{x} &= -\sigma x+\sigma y, \qquad & \sigma &= 10,\\ \dot{y} &= -xz+rx-y, \qquad & r &= 28,\\ \dot{z} &= xy-bz, \qquad & b &= \tfrac{8}{3}. \end{aligned}$$

Lorenz (1963) studied the time development numerically (figure 1). His result was that the trajectories exhibit the following behaviour:

(i) Trajectories corresponding to nearby initial points separate exponentially in time, a characteristic of chaotic behaviour.

(ii) Nevertheless the trajectories seem for a large set of initial points to approach a set independent of the initial point – an *attractor*.

2. *The Hénon map*. Although seemingly fairly simple, the dynamics of the Lorenz equations appeared too complicated to allow a mathematical description. The astronomer Hénon (1976) suggested a two-dimensional discrete dynamical system as the least complicated model problem, which would exhibit the same features as the Lorenz system. He considered the map $T\colon \mathbb{R}^2 \mapsto \mathbb{R}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1+y-ax^2 \\ bx \end{pmatrix},$$

chose the parameters $a = 1.4$, $b = 0.3$, and plotted a large number of iterates. The

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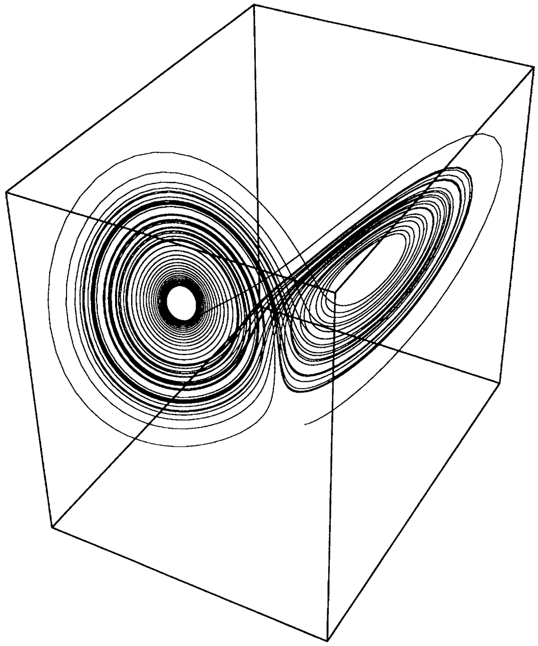

Figure 1.

result is the now famous picture (figure 2), which exhibits a fractal line structure on successive enlargements.

Note that the study of a three-dimensional flow such as the Lorenz system can be reduced to that of a two-dimensional map through the idea of a first return (or Poincaré map) to a hyperplane.

2. Attractors

The Hénon map has an attractor. Several definitions of attractors have been suggested. Let f be a diffeomorphism of a manifold M. We use the notation $f^n = f \circ \cdots \circ f$ for the nth iterate of f. Recall that the *omega limit set* $\omega(x)$ is the set of all accumulation points of the orbit $\{f^n(x)\}_{n=0}^{\infty}$.

The following definition of attractor was proposed by Charles Conley:

Definition 2.1. The set Λ is an *attractor* for the diffeomorphism f if there is an open set $U \supset \Lambda$ such that $f(\bar{U}) \subset U$ and $\Lambda = \bigcap_{n=0}^{\infty} f^n(\bar{U})$. The *basin of attraction* for Λ is $\bigcup_{n=0}^{\infty} f^{-n}(U)$.

However, this definition does not exclude the possibility that say Λ may be written as the disjoint union of two pieces Λ_1 and Λ_2 and that there are two open sets $U_1 \supset \Lambda_1$ and $U_2 \supset \Lambda_2$ such that $f(\bar{U}_1) \subset U_1$ and $f(\bar{U}_2) \subset U_2$ and $\Lambda_i = \bigcap_{n=0}^{\infty} f^n(\bar{U}_i)$, $i = 1, 2$. We therefore need the following notion.

Definition 2.2. Λ is an *indecomposable* or *topologically transitive* attractor if there is a point $x_0 \in \Lambda$ such that $\Lambda = \overline{\bigcup_{n=0}^{\infty} f^n(x_0)}$.

There are other definitions of attractors, most notably those suggested by Ruelle (1981) and Milnor (1985), which allow for more general attractors. Milnor's definition is as follows:

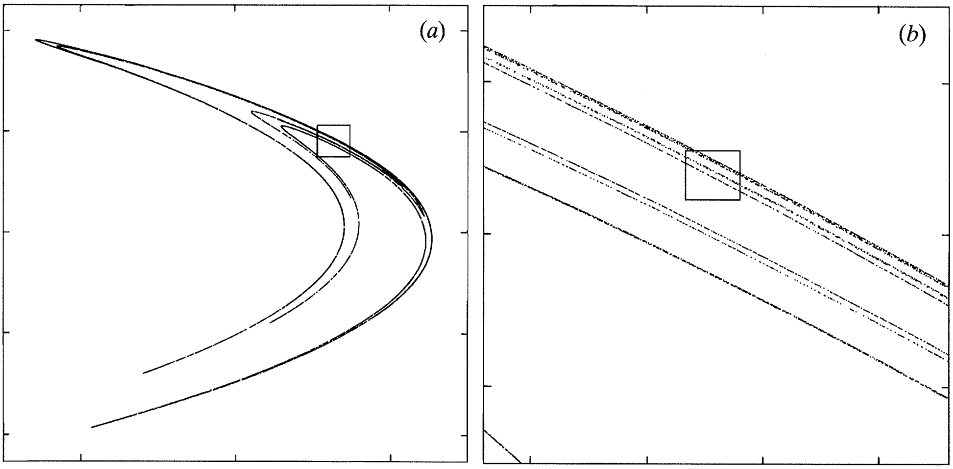

Figure 2.

Definition 2.3 (Milnor). A closed subset $A \subset M$ is an attractor if it satisfies two conditions.

1. The *realm of attraction* $\rho(A)$, consisting of all points $x \in M$ for which $\omega(x) \subset A$ is of positive measure.
2. There is no strictly smaller closed set $A' \subset A$ so that $\rho(A')$ and $\rho(A)$ coincide up to a set of measure 0.

The first condition says that there is a positive probability that a randomly chosen point will be attracted to A, and the second is a minimality conditions to ensure that A does not contain any superfluous points.

Hénon, however, verified that his attractor for his parameter values was an attractor in the sense of Conley. The question whether it is a topologically transitive attractor is much harder as we shall see.

3. Metric description of attractors

To understand the statistical properties of an orbit $\{x_j\}_{j=0}^{\infty}$, $x_j = f^j(x_0)$, it is natural to form the Birkhoff sums

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{x_j},$$

where δ_a denotes the Dirac measure at a. By taking subsequences, we obtain that a subsequence converges weak-$*$ to a limiting measure μ^*. It is easy to verify that μ^* is an invariant measure, i.e.

$$\mu^*(f^{-1}E) = \mu^*(E)$$

for all Borel sets E. Let us from now on drop the $*$ and let μ denote the invariant measure. By Birkhoff's ergodic theorem for a.e. x_0 $[\mu]$

$$\mu_n^{(x_0)} = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{x_j}$$

has a weak-$*$ limit.

If moreover μ is ergodic then $\mu_n^{(x_0)} \to \mu$ as $n \to \infty$ for a.e. x_0 $[\mu]$. Such a point x_0 is called a generic point for the measure μ. Only those limiting measures are really interesting that have the property that a large set of initial points are generic.

This is the background for the following definition due to Ruelle:

Definition 3.1. μ is a *physical measure* if $\mu_n^{(x_0)} \to \mu$ as $n \to \infty$ for a set of initial points x_0 of positive Lebesgue measure.

Suppose now that f has a stable periodic orbit, i.e. an orbit $\{z_j\}_{j=0}^{p-1}$ such that $z_0 = f^p(z_0)$, and that the spectrum of $Df^p(z_0)$, $\sigma(Df^p(z_0)) \subset \{|z| < 1\}$ or equivalently that $\|Df^p(z_0)\| < 1$ for some suitably chosen norm. Then it is clear that there are neighbourhoods $\{U_i\}_{i=0}^{p-1}$ of the points of the orbit such that $\forall x_0 \in \bigcup_{i=0}^{p-1} U_i$,

$$\mu_n^{(x_0)} \to \frac{1}{p} \sum_{j=0}^{p-1} \delta z_j, \quad \text{as} \quad n \to \infty.$$

An alternative definition of an attractor is then to *define an attractor as a physical measure*. This definition is in fact quite close to that of Milnor. The minimal requirement on a strange attractor is then that it is a physical measure different from an average of point masses along a stable periodic orbit.

Example 3.1. Consider the one-dimensional map $f\colon [-1, 1] \circlearrowright$ given by $f(x) = 1-2x^2$. There is an explicit change of variables so that the map in the new variables is given by

$$\tilde{f}(y) = \phi^{-1} \circ f \circ \phi(y) = 1-2|y|.$$

The map $y \mapsto 1-2|y|$ has the property that Lebesgue measure is invariant. The change of variables transports Lebesgue measure into a density $c/\sqrt{(1-x^2)}\,dx$, which is invariant for the original map $x \mapsto 1-2x^2$. Note that from the change of variables we can conclude that for all $\epsilon > 0$ there is $n_0 = n_0(\epsilon)$, a constant $K = K(\epsilon)$ and a set E_ϵ, satisfying $\text{Leb}(\complement E_\epsilon) < \epsilon$, so that

$$|Df^n(x)| \geqslant K(\epsilon) 2^{n(1-\epsilon)} \quad \forall n \geqslant n_0(\epsilon) \quad \forall x \in E_\epsilon. \tag{3.1}$$

We also have that for all $x \neq \pm 1$ there is a constant $A(x)$ so that

$$|Df^n(x)| \leqslant A(x) 2^n. \tag{3.2}$$

The estimates (3.1) and (3.2) are *non-uniform*. There are always points that return arbitrarily close to 0. However, from (3.1) and (3.2) it follows that

$$\lim_{n\to\infty} \frac{1}{n} \log |Df^n(x)| = \log 2, \quad \text{for} \quad \text{a.e. } x,$$

a concrete example of Oseledets theorem (see Theorem 5.1 below).

4. The uniformly hyperbolic case

Suppose that M is a riemannian manifold, f is a diffeomorphism and that Λ is a compact invariant set.

Definition 4.1. f is *uniformly hyperbolic* on Λ if there is a splitting of the tangent bundle of Λ into two Df-invariant subbundles $T\Lambda = E^u \oplus E^s$ and that there are constants $C > 0$ and $\lambda > 1$ such that for all $n \geqslant 0$

$$|Df_x^{-n} v| \leqslant C\lambda^{-n} |v|, \quad \forall x \in \Lambda \;\; \forall v \in E^u(x),$$
$$|Df_x^{n} v| \leqslant C\lambda^{-n} |v|, \quad \forall x \in \Lambda \;\; \forall v \in E^s(x).$$

f is called *Anosov* if f is uniformly hyperbolic on the entire manifold.

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Example 4.1. The map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \pmod 1$$

viewed as a map $f\colon \mathbb{T}^2 \circlearrowright$ is the canonical example of an Anosov diffeomorphism.

Definition 4.2. The invariant set Λ is called an *Axiom-A* basic set if

(1) $f|_\Lambda$ is uniformly hyperbolic;
(2) there is a neighbourhood U of Λ such that $\Lambda = \{x \in U : f^n(x) \in U\ \forall n \in \mathbb{Z}\}$;
(3) $f|_\Lambda$ has a dense orbit.

For a general diffeomorphism $f\colon M \circlearrowright$ define the *wandering set* W as the set of points $x \in M$, such that there is a neighbourhood $U \ni x$ such that $f^n(U) \cap U = \varnothing, \forall n \geqslant 1$. The complement of the wandering set is $\Omega(f)$, the *non-wandering set*.

Definition 4.3. A diffeomorphism $f\colon M \circlearrowright$ satisfies Axiom-A if there is a decomposition of the non-wandering set $\Omega(f)$ as a disjoint union $\Omega(f) = \bigcup_{i=1}^n \Lambda_i$, where Λ_i are Axiom-A basic sets.

(This definition is usually a theorem – Smale's spectral decomposition – if one uses the usual definition of Axiom-A diffeomorphisms.)

Certain of the Λ_i may be attractors in the sense of Conley.

5. Lyapunov exponents

If A is an $n \times n$ matrix there is decomposition into invariant subspaces

$$\mathbb{R}^n = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

so that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |A^n v| = \lambda_i, \quad \forall v \in E_i.$$

(This follows easily from a representation of A in Jordan's normal form.)

A nonlinear version of this result is the following theorem due to Oseledets.

Theorem 5.1. *Suppose $f\colon M \circlearrowright$ is a C^1-diffeomorphism with invariant measure μ. Then for a.e. $x\,[\mu]$, there is a splitting of the tangent space*

$$TM_x = E_1(x) \oplus \cdots E_k(x)$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |Df_x^n v| = \lambda_i(x), \quad \forall v \in E_i, 1 \leqslant i \leqslant k.$$

(We have to make the technical assumptions that $\int \log^+ |Df|\,\mathrm{d}\mu$, $\int \log^+ |Df^{-1}|\,\mathrm{d}\mu < \infty$.)

If moreover μ is ergodic $\lambda_i(x) \equiv \lambda_i$ is constant a.e. $[\mu]$.

6. Stable and unstable manifolds

We define the *stable manifold* at x as

$$W^s(x) = \{y \in M : \lim_{n\to\infty} d(f^n x, f^n y) = 0\}$$

and the *unstable manifold* at x as

$$W^u(x) = \{y \in M: \lim_{n\to-\infty} d(f^n x, f^n y) = 0\}.$$

If x_0 is a hyperbolic fixed point, i.e. a fixed point such that, $\sigma(Df(x_0)) \cap \{|z| = 1\} = \varnothing$, there is a neighbourhood U of x_0 such that $W^u_{\text{loc}}(x_0) = \bigcap_{n\geqslant 0} f^n(U)$ is an immersed submanifold and $W^u(x_0) = \bigcup_{n\geqslant 0} f^{-n}(W^u_{\text{loc}}(x_0))$.

A similar statement is obtained for $W^s(x_0)$ by reversing the arrow of time.

The theory of generalized stable and unstable manifolds in the uniformly hyperbolic case is due to Hirsh, Pugh and Shub and a good reference is Shub (1987). A corresponding theory in the non-uniformly hyperbolic case is due to Pesin (1977).

Theorem 6.1 (Pesin's Stable Manifold Theorem). *If μ is an invariant measure for the diffeomorphism f such that for a.e. x $[\mu]$, $|\lambda_i(x)| \neq 1$ for all i, then for a.e. x $[\mu]$ $W^u(x)$ and $W^s(x)$ are immersed submanifolds.*

One can then form the unstable foliation $\mathscr{F}^u = \bigcup_{x\in\Lambda} W^u(x)$ and the stable foliation $\mathscr{F}^u = \bigcup_{x\in\Lambda} W^s(x)$. For an exposition of the proof see Fahti *et al.* (1983) or Pugh & Shub (1989).

7. Sinai–Ruelle–Bowen measures

Definition 7.1. A Sinai–Ruelle–Bowen measure is an invariant probability measure μ such that μ can be represented (disintegrated) as

$$\mu = \int_A \mu_\alpha(\cdot)\, m(\mathrm{d}\alpha)$$

$$\operatorname{supp}(\mu_\alpha) \subset S_\alpha = W^u(x_\alpha) \subset \mathscr{F}^u,$$

where $\mathrm{d}\mu_\alpha = \phi_\alpha\, \mathrm{d}x$ is absolutely continuous with respect to the induced riemannian measure.

An example of a measure of this type is the following. Let $\Lambda = I \times C$, where C is a Cantor set, say the usual triadic Cantor set, and let $\mu = m \times \mu_c$, where m is Lebesgue measure on I, and μ_c is the Cantor measure associated with the set C.

SRB-measures where first constructed for Anosov mappings by Sinai and for Axiom-A attractors by Ruelle and Bowen.

Theorem 7.1. *If Λ is a Axiom-A attractor it has a unique SRB-measure μ. For a.e. x in the basin of attraction of Λ*

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j x} \to \mu.$$

One of the aims of the present paper is to describe to which extent this can be carried over to the non-uniformly hyperbolic situation.

Theorem 7.2 (Pesin). *Let f be a C^1-diffeomorphisms with an invariant SRB-measure μ such that $|\lambda_i(x)| \neq 1$ $\forall i$ a.e. $[\mu]$. Then there exist at most countably many sets $\{\Lambda_i\}_{i=1}^\infty$ with $\mu(\Lambda_i) > 0$ and $\mu(\bigcup \Lambda_i) = 1$ such that $(f|_{\Lambda_i}, \mu|_{\Lambda_i})$ is ergodic. For each $\mu_i = \mu|_{\lambda_i}$, there is X_i of positive Lebesgue measure such that*

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j x} \to \mu_i \quad \forall x \in X_i.$$

For an exposition of the proof see Pugh & Shub (1989).

The Pesin theory is very satisfactory except for that there is one thing missing – the construction of SRB-measures. This construction is now well-known in the Axiom-A situation but most 'real-life' dynamical systems are non-uniformly hyperbolic.

8. The one-dimensional case

We consider the quadratic family of maps $f_a(x) = 1 - ax^2$, $0 < a \leqslant 2$. For this range of parameters f_a maps the unit interval $[-1, 1]$ into itself and we have the following celebrated theorem of Jakobson (1981) concerning chaotic behaviour.

Theorem 8.1 (Jakobson). *There is a set $A \subset (0, 2]$ of parameters of positive Lebesgue measure so that $\forall a \in A$*

(1) $|Df_a^n(f(0))| \geqslant e^{cn}$, $\forall n \geqslant 0$; $c = c(A) > 0$;
(2) f_a *has an absolutely continuous invariant measure* μ_a;
(3) f_a *has positive Lyapunov exponent a.e.* $[\mu_a]$.

There are several approaches to this result – apart from Jakobson's original – by Benedicks & Carleson (1985, 1991), Rychlik (1988), Yoccoz (1990) and others. In Benedicks & Carleson's approach initially a parameter set A' is chosen as

$$A' = \{a : |f_a^j(0))| \geqslant e^{-\alpha j} \quad \forall j \geqslant 1\}.$$

One verifies that A' is of positive Lebesgue measure. The final set A is then chosen as a subset of A' – still of positive measure – such that the critical orbit has a 'typical' statistical behaviour relative to the critical point 0. The parameter values in A' are characterized by a slow approach rate to the critical point 0. This may be compared with Misiurewics' condition $|f_a^j(0)| \geqslant \delta, \forall j \geqslant 0$, i.e. the critical orbit avoids a fixed neighbourhood of the critical point. The set of a such that Misiurewics' condition is satisfied is of Lebesgue measure 0.

For a sufficiently close to 2, f_a is essentially expanding outside a fixed neighbourhood $V = (-\delta, \delta)$ of 0. For $x \in V$ there is $p = p(x)$ so that

$$|Df^p(x)| = |2ax| \cdot |Df^{p-1}(f(x))| \geqslant e^{c'p},$$

where c' is a fixed constant. For $j = 0, 1, \ldots, p$, $f^j(x)$ follows $f^j(0)$ closely and $Df^{p-1}(f(x))$ compensates for the small factor $|2ax|$.

The invariant measure is constructed by weak limits of $\mu_n = n^{-1} \sum_{j=0}^{n-1} f_*^j m$, where m is Lebesgue measure and $(f_*^j m)(E) = m(f^{-j} E)$.

Remark 1. The set $B = \{a : f_a$ has a stable periodic orbit$\}$ has for a long time been believed to be dense in $(0, 2)$. This result is now claimed in Świątek (1992).

Remark 2. The density ϕ_a of μ_a for $a \in A$ has an estimate

$$\phi_a(x) \leqslant C + \sum_j \frac{c_j}{\sqrt{|x - f^j(0)|}}.$$

This is a result of Young (1992).

Remark 3. Other behaviour appears for $a \in (0, 2)$. For the limiting parameter a_∞ of the Feigenbaum bifurcations there is a physical measure, which is purely singular with respect to Lebesgue measure. Its support is an attractor in the sense of Milnor but not in the sense of Conley. Hofbauer & Keller (1990) proved the surprising result that there are parameters value a such that there is a set of initial points X of positive Lebesgue measure such that $n^{-1}\sum_{j=0}^{n-1}\delta_{f^j(x)}\rightarrow\delta_y$, where y is the unstable fixed point.

A natural conjecture is that the union of the points, where f_a has an absolutely continuous invariant measure and the points where f_a has a stable periodic orbit, is a subset of $(0,2)$ of full Lebesgue measure.

9. The Hénon map

For the Hénon family

$$\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}1+y-ax^2\\bx\end{pmatrix},\quad 0<a<2,\quad b>0,$$

we have the following result:

Theorem 9.1 (Benedicks & Carleson 1991). *There is a set Δ of parameters (a,b) of positive two-dimensional Lebesgue measure such that* $\forall(a,b)\in\Delta$:

(1) *there is a compact invariant set $\Lambda=\Lambda_{a,b}$ and an open neighbourhood $U\supset\Lambda$ so that $\forall z\in U$, $T^jz\rightarrow\Lambda$ as $j\rightarrow\infty$; the attractor Λ is the closure of the unstable manifold $W^u(\hat{z})$, where $\hat{z}$ is the fixed point of T in the first quadrant;*

(2) *there is a point z_0 such that the orbit $\{T^jz_0\}_{j=0}^{\infty}$ is dense in Λ and $|DT^j(z_0)\binom{0}{1}|\geqslant e^{cj}$, $j=0,1,\ldots,$ for some $c>0$.*

Benedicks & Young (1993) proved furthermore

Theorem 9.2. *For $(a,b)\in\Delta$ as in Theorem 9.1,*

(1) *$T|_\Lambda$ has a unique SRB-measure such that* $\operatorname{supp}(\mu)=\Lambda$;

(2) *there is a set $X\subset U$, of positive Lebesgue measure such that*

$$\frac{1}{n}\sum_{j=0}^{n-1}\delta_{T^jz}\rightarrow\mu\quad\forall z\in X.$$

An interesting open problem is to determine whether (2) holds for a.e. $z\in U$ with respect to Lebesgue measure as is the case for Axiom-A attractors.

Remarks on the proofs. In the one-dimensional case the main idea is to control the orbit of the critical point 0. In the Hénon case the critical point is replaced by a critical set $\mathscr{C}\subset W^u(\hat{z})$ having the following properties

(1) $\mathscr{C}$ is countable;

(2) $\mathscr{C}\subset(-\delta,\delta)\times\mathbb{R}$, where $\delta>0$ and small;

(3) for all $z\in\mathscr{C}$, $|DT_z^ju|\leqslant(5b)^j$ $\forall j\geqslant0$, where u is the tangent vector of W at z.

Intuitively, the points of $\mathscr{C}$ have the property that they are mapped on 'sharp turns' of $W^u(\hat{z})$ (figure 3).

The main objective is to prove that $w_j=DT_{z_0}^j\binom{0}{1}$ satisfies $|w_j|\geqslant e^{cj}$ $\forall z_0\in\mathscr{C}$ $\forall j\geqslant0$. (Compare the result in the one-dimensional situation that $|Df_a^j(1)|\geqslant e^{cj}$.) When the point z_n returns to $(-\delta,\delta)\times\mathbb{R}$, the vector $w_n=DT_{z_0}^n\binom{0}{1}$ will essentially align with the stable direction and is contracted heavily. However, at the return at time n we associate to z_n another critical point $\tilde{z}_0$, so that (i) $DT_{\tilde{z}_0}^j\binom{0}{1}|\geqslant e^{cj}$ for $0\leqslant j<p$; (ii) $\{\tilde{z}_j\}_{j=1}^p$ 'controls' $\{z_{n+j}\}_{j=1}^p$ and the associated derivatives.

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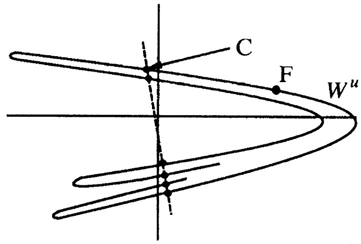

Figure 3.

The basic idea is then that the vector w_n is initially contracted during the 'fold period' but that we obtain the same type of compensation of the loss in size of the vector w_n during the 'bound period' $1, \ldots, p$ as in the one-dimensional case. We have to exclude parameters corresponding to each critical point to ensure that $\text{dist}(T^j(z_0), \mathscr{C}) \geqslant e^{-\alpha j}\, \forall j \geqslant 1$. It is important to allow an exponential approach rate in order that the excluded set corresponding to each critical point and time $\geqslant n$ should be exponentially small in n. This is necessary since the number of critical points, which are relevant at time n grows exponentially. ∎

It is important that the critical set is only defined for the parameters $(a, b) \in \varDelta$. The proof goes by induction and at the Nth stage of the induction we have a finite approximation $\mathscr{C}_N$ of the eventual critical set $\mathscr{C}$, i.e. for $z \in \mathscr{C}_N$, (3) is only satisfied for $j \leqslant N$.

Let us explain how this successive definition of the $\mathscr{C}_N$ starts by a concrete computation. Note that the first segment of the unstable manifold through the fixed point has a representation as an approximate parabola

$$x = 1 - a(y/b)^2 + r(y),$$

where $\|r\|_{C^1} = \mathscr{O}(b)$. We conclude that the slopes $\tau(x)$ of the upper and lower branches of the unstable manifold satisfy $\tau(x) = \mp(b/2a)\ ((1-x)/a))^{-1/2} + \mathscr{O}(b^2)$ respectively. One verifies that the most contractive direction $s(x)$ of $DT(z)$ only depends on x and satisfies $s(x) = 2ax + \mathscr{O}(b^2)$. The equation $\tau(x) = s(x)$ has the two solutions $(x, y) = (\mp b/4a^2, \pm b/a^{1/2}) + \mathscr{O}(b^2)$ on $W^u_{\text{loc}}(\hat{z})$, which are located in the second and fourth quadrant respectively and these two points constitute two initial approximate critical points, see figure 3.

In view of Pesin theory the critical set corresponds to tangencies between the unstable manifold $W^u(\hat{z})$ and local stable manifolds. In the uniformly hyperbolic case the angles between local unstable and stable manifolds are uniformly bounded below. In some sense the critical set isolates the non-uniform hyperbolic behaviour and the main idea of the proof is to control its dynamics.

It is therefore interesting to understand the properties of the critical set. In a master's thesis at the Royal Institute of Technology, G. Ryd has proved that $\mathscr{C}$ is located on a curve, which in local C^2 coordinate patches may be given as the graph of a C^α-function, $0 < \alpha < 1$. It does not seem possible to make this curve C^1.

10. Dimension of the attractor

Let us recall the following two theorems. The first is an extension of a result of Pesin. The case needed here is due to Ledrappier, but see Ledrappier & Young (1985) for considerable improvements.

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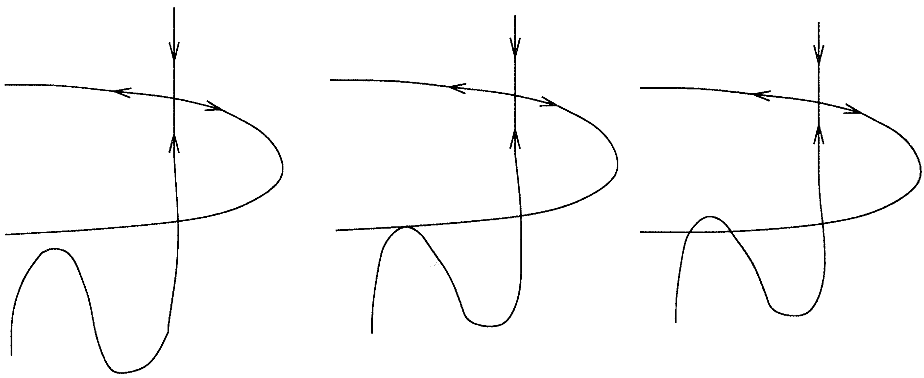

Figure 4. From left to right: $\mu < 0$, $\mu = 0$, $\mu > 0$.

Theorem 10.1. *If μ is an ergodic SRB-measure then the metric entropy h_μ satisfies*
(10.1)

$$h_\mu = \sum_i \lambda_i^+,$$

i.e. it is the sum of the positive Lyapunov exponents.

Ledrappier & Young proved that Pesin's formula (10.1) is true if and only if μ is an SRB measure.

The second theorem is due to Young.

Theorem 10.2. *If f is a C^2-diffeomorphism of a two-dimensional manifold with an ergodic invariant measure μ with Lyapunov exponents $\lambda_1 > 0 > \lambda_2$. Then the Hausdorff dimension of the support of the measure μ defined as $HD(\mu) = \inf_{\mu(X)=1} HD(X)$ satisfies*

$$HD(\mu) = h_\mu(f)\left[\frac{1}{\lambda_1} + \frac{1}{|\lambda_2|}\right].$$

In the Hénon case μ is SRB by Theorem 9.2 and from Theorem 10.1 we conclude that $h_\mu = \lambda_1$ and hence by Theorem 10.2

$$HD(\mu) = \lambda_1\left(\frac{1}{\lambda_1} + \frac{1}{|\lambda_2|}\right) = 1 + \frac{\lambda_1}{\lambda_1 + \log(1/b)},$$

which is > 1 but $\to 1$ as $b \to 0$.

11. Homoclinic bifurcations

Consider a one-parameter family of diffeomorphisms f_μ, $-\delta < \mu < \delta$, of a two-manifold such that f_μ has a hyperbolic fixed point x_μ and that the eigenvalues λ_1 and λ_2 at this fixed point satisfy $|\lambda_1| \cdot |\lambda_2| < 1$ (the dissipative case). Suppose that for $\mu = 0$ the stable manifold $W^s(x_\mu)$ and the unstable manifold $W^u(x_\mu)$ have a (homoclinic) tangency. Furthermore assume that this homoclinic intersection does not exist for $\mu < 0$ (figure 4). This situation is called a homoclinic bifurcation and is the topic of the recent book by Palis & Takens (1993). Arbitrarily close to a homoclinic bifurcation there are a host of phenomena, e.g. hyperbolic behaviour (Palis, Takens, Yoccoz) and maps with infinitely many sinks (Newhouse). Palis proposed the very general conjecture that an arbitrary C^2-diffeomorphism may be approximated arbitrarily well either by uniformly hyperbolic maps or by maps exhibiting a homoclinic bifurcation.

Mora & Viana (1994) have recently proved that in the case of a generic homoclinic bifurcation there is a positive Lebesgue measure set of parameters μ for which f_μ has a Hénon type strange attractor. The starting point of their proof is the following:

Note first that the Hénon family of maps after a change of coordinates

$$x_1 = x, \quad y_1 = y/\sqrt{b}$$

may be written

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \to \begin{pmatrix} 1-ax_1^2 \\ 0 \end{pmatrix} + \sqrt{b}\begin{pmatrix} y_1 \\ x_1 \end{pmatrix}.$$

Suppose that $f_\mu \in C^3$. One of the main ideas in the book of Palis & Takens is that one can choose a small neighbourhood of the homoclinic bifurcation and a high iterate N and find a linear change of variables Λ both in the coordinates and the parameters such that the map $\phi_N = \Lambda^{-1} \circ f^N \circ \Lambda$ may be written

$$\phi_N = \begin{pmatrix} 1-ax^2 \\ 0 \end{pmatrix} + \psi_N,$$

where the diffeomorphism $\|\psi_N\|_{C^3} = b^t$, for some $t > 0$, $b = (|\lambda_1 \lambda_2|)^N$.

Mora & Viana then carries out the same program as in Benedicks & Carleson (1991) but with more complicated estimates. Note that their attractor is located in the orbit of a small neighbourhood of the homoclinic bifurcation. One cannot conclude, however, that there is only one transitive attractor as in Benedicks & Carleson.

The computer pictures seem to indicate that there is a homoclinic bifurcation for the Hénon family close to the classical parameters $a = 1.4$ and $b = 0.3$. This has also recently been rigorously verified by Fornaess. Therefore one may hope that the theorem of Mora & Viana should apply and give a strange attractor close to the classical parameters. However, in view of the remark above this is not sufficient to make the conclusion that there is a single ergodic attractor equal to $\overline{W^u(\hat{z})}$.

12. Random perturbations and statistical behaviour

It is natural to consider what happens to a dynamical systems when one allows random perturbations in each iteration. A typical model problem is the following.

Let Y_n be independent, identically distributed random variables with a uniform distribution on $[-\epsilon, \epsilon]$. Now define a Markov chain $\{X_n\}_{n=0}^{\infty}$ by

$$X_{n+1} = 1 - aX_n^2 + Y_n, \quad X_0 \equiv x_0 = \text{const.},$$

where $0 < a < 2$. For a sufficiently close to 2, there is always a unique stationary probability measure P_ϵ^a for the Markov chain. A natural question is how this probability measure is related to an invariant measure μ_a for the unperturbed map $f_a(x) = 1 - ax^2$. The following is a consequence of a result in Benedicks & Young (1991).

Theorem 12.1 (Benedicks & Young). *Consider the map* $f_a(x) = 1 - ax^2$, *where* $a \in A$, *the set of parameters defined in Benedicks & Carleson* (1985). *Then* f_a *has an absolutely continuous invariant measure* μ_a *and the stationary measures* P_ϵ^a *of the Markov chain* $\{X_n\}_{n=0}^{\infty}$ *tend weakly to* μ_a *as* $\epsilon \to 0$.

Hence the distributions of a typical perturbed and unperturbed orbit are essentially the same. This at least partly explains why the results of the computer experiments seem to be correct in spite of intrinsic round off errors.

For $a \in A$, again the same set as in Benedicks & Carleson (1985), one may view an orbit $\{x_n\}_{n=0}^{\infty}$ of the map f_a as a stationary stochastic process with invariant probability measure μ_a. It is natural to ask how 'independent' x_n and x_m are when $|n-m|$ is large. A result in this direction is the following theorem on the decay of correlation:

Theorem 12.2 (Young 1992). *Consider $f(x) = f_a(x)$ and the corresponding invariant measure $\mu = \mu_a$ for $a \in A$, the parameter set of Benedicks & Carleson* (1991). *Suppose that ϕ and ψ are functions of bounded variation such that*

$$\int \phi(x)\,\mathrm{d}\mu(x) = \int \psi(x)\,\mathrm{d}\mu(x) = 0.$$

Then there are constants $c > 0$ and $C > 0$ such that

$$\left| \int \phi(f^n(x))\,\psi(f^m(x))\,\mathrm{d}\mu(x) \right| \leqslant C\mathrm{e}^{-c|n-m|}.$$

Results similar to Theorem 12.1 and Theorem 12.2 have existed for some time in the uniformly hyperbolic setting. In the non-uniform hyperbolic case in two variables the corresponding results are still unknown.

13. Open problems

Several open problems have already been described above but I wish to finish by mentioning some problems, which indicate in which direction one would like to pursue this subject.

A major open problem is to extend the two-dimensional theory in the dissipative (area shrinking) case described above to an area preserving case. A famous model dynamical system is the standard map family f_K. These are maps of the two-torus $\mathbb{T}^2$ defined by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - y + K \sin(2\pi x) \\ x \end{pmatrix} \quad (\mathrm{mod}\ 1).$$

The main conjecture is that there is a set of positive one-dimensional measure of parameters K such that there is a positive Lebesgue measure set of initial points with positive Lyapunov exponent.

One of the main new difficulties here, in contrast to the case of Hénon maps, is that one has to deal with a critical set which is essentially one dimensional. It is no longer possible to enforce that the orbit of a point of the critical set avoids the critical set.

A new result in the opposite direction (which however does not contradict the conjecture) is the following recent result due to Duarte (1993).

Theorem 13.1. *There is a sequence of parameters $K_n \to \infty$ such that for f_{K_n} in the standard map family, the closure of the union of the elliptic fixed points E_n has the property that the Hausdorff dimension $HD(E_n) \to 2$ as $n \to \infty$.*

Finally, the problem of the Lorenz equations still remains unsolved in spite of major efforts. The situation in this case seems, however, to be different from that of the Hénon map. Numerical evidence suggest that the Poincaré map, although discontinuous, is piecewise uniformly hyperbolic. This could in principle be proved by sufficiently good numerical estimates. Assuming this is true, the system can be studied satisfactorily by symbolic dynamics (Guckenheimer & Williams 1979). While it is more complicated than the Axiom-A case, this picture is much simpler than that of the Hénon map. Another relevant reference is the book by Sparrow (1982) on the Lorenz equations.

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